9.1

Basic Combinatorics

The Importance of Counting:

In how many different ways can three distinguishable objects be arranged in order?

Let's call the objects A, B and C. The different orderings are: ABC, ACB, BAC, BCA, CAB, and CBA.

Using a tree diagram can help with this.

We can also think of it this way: there are 3 ways to choose the first object, 2 ways to choose the second object and only one way to choose the last so, there are $3 \cdot 2 \cdot 1 = 6$ ways to order the objects.

Multiplication Principle of Counting -- If a procedure has a sequence of stages

$S_1, S_2, \ldots S_n$ *and if* S_1 *can occur in* r_1 *ways,*

S_2 *can occur in* r_2 *ways,*

$\vdots$

S_n *can occur in* r_n *ways*

then the number of ways that the procedure P can occur is the product

$r_1 r_2 \cdots r_n$

Ex. Using the Multiplication Principle

The Utah license plate consists of 3 letters and 3 numbers. Find the number of different license plates that can be formed:

a) if there are no restrictions on what letters or number that can be used;
b) if no letter or digit can be repeated.

Solution:

a) If there are no restrictions then we can fill in the first three letters 26 ways and the numbers 10 ways, so $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$ ways license plates can be made.

b) If no letter or number can be repeated then the first letter can be chosen 26 ways the second 25 and the third 24. The first number can be chosen 10 ways the second 9 and the third 8 so, $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 = 11{,}232{,}000$ ways.

Permutations of an n-set:

There are n! permutations of an n-set.

Distinguishable Permutations:

Count the number of different "words" that can be formed using the letters in each word.

a) BINGHAM b) MINERS c) FOOTBALL

Solution:

a) Each permutation of the 7 letters form a different word therefore, there are 7! = 5040 ways.
b) This is the same with no repeating letters therefore there are 6! = 720 ways.
c) Here we have two letters that repeat, the O's and L's each repeat twice. To find the distinguishable permutations we must correct the over counting by dividing therefore there are $\frac{8!}{2!2!}$ ways .

<u>Distinguishable Permutations</u>

There are n! distinguishable permutations of an n-set containing n distinguishable objects.

If an n-set contains n_1 objects of a first kind, n_2 objects of a second kind, and so on, with $n_1+n_2+...+n_k=n$, then the number of distinguishable permutations of the n-set is:

$$\frac{n!}{n_1!n_2!n_3!...n_k!}.$$

Permutations – order matters (abc is not same as bac)

The number of permutations of n objects taken r at a time is denoted $_nP_r$ and is given by

$$_nP_r = \frac{n!}{(n-r)!} \quad for\ 0 \le r \le n$$

$$If\ r > n, then\ _nP_r = 0$$

Note: 0! is defined as 1 so, 0! = 1.

Counting Permutations:

Evaluate each expression without a calculator.

a) ${}_6P_4$ b) ${}_{11}P_3$ c) ${}_nP_3$

Solution:

a) By the formula ${}_6P_4 = \frac{6!}{(6-4)!} = \frac{(6\cdot5\cdot4\cdot3\cdot2!)}{2!} = 360$.

b) Here we can use the formula again, but we can also use the Multiplication Principle directly. We have 11 objects and 3 blanks to fill: ${}_{11}P_3$ = $11\cdot10\cdot9 = 990$.

c) Using the Multiplication Principle here is also easier. We have n objects and 3 blanks to fill so: ${}_nP_3$ = n(n – 1)(n – 2).

Example:

There are 16 players on a baseball team, and only 9 can play at a time. In how many ways can the coach play the members on the team in each position?

Solution: This is a permutation (order matters)

b) ${}_{16}P_9 = \frac{16!}{(16-9)!} = \frac{(16\cdot15\cdot14\cdot13\cdot12\cdot11\cdot10\cdot9\cdot8\cdot7!)}{7!} = 4{,}151{,}347{,}200$

Combinations – order does not matter (abc = cba)

The number of combinations of n objects taken r at a time is denoted $_nC_r$ and is given by

$$_nC_r = \frac{n!}{r!(n-r)!} \quad for \ 0 \le r \le n$$

$$If \ r > n, then \ _nC_r = 0$$

Example:

In the Miss America pageant, 51 contestants must be narrowed down to 10 finalists who will compete on national television. In how many possible ways can the ten finalists be selected?

This is a combination because order does not matter so,

$$_{51}C_{10} = \frac{51!}{10!41!} = 12,777,711,870.$$

Distinguishing Combinations from Permutations

In each of the following scenarios, tell whether permutations or combinations are being described.

a) A president, vice-president, and secretary are chosen from a group of 25 candidates.

b) A cook chooses 5 potatoes from a bag of 12 potatoes to make a salad.

c) A teacher makes a seating chart for 35 students in a classroom with 37 desks.

Formula for counting subsets of an n-set

There are 2^n subsets of a set with n objects (including the empty set and the entire set).

Example:

The Pie offers customers any combination of up to 10 different toppings: pepperoni, mushroom, sausage, onion, green pepper, bacon, pineapple, ham, olives, and anchovies. How many different pizzas can be ordered

a) If we can choose any three toppings?

b) If we can choose any number of toppings (0 – 10)?

Solution:

a) Order does not matter so this is a combination ${}_{10}C_3 = \frac{10!}{3!7!} = 120$.

b) This example uses subsets of a set so we have $2^n = 2^{10} = 1024$.